PSYCHOMETRIKA—VOL. 27, NO. 1
MARCH, 1962

A FUNCTIONAL EQUATION ANALYSIS OF TWO LEARNING MODELS*

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One-absorbing barrier random walks arising from Luce's nonlinear beta model for learning and a linear commuting-operator model (called the alpha model) are considered. Functional equations for various statistics are derived from the branching processes defined by the two models. Solutions to general functional equations, satisfied by statistics of the alpha and beta models, are obtained. The methods presented have application to other learning models.

The two-response, two-event, path-independent, contingent version of a number of stochastic models for learning is given by the equations

$$p_{n+1} = \begin{cases} Q_1 p_n & \text{with probability } p_n \\ Q_2 p_n & \text{with probability } (1 - p_n), \end{cases} \tag{1}$$

where Q_1 and Q_2 represent transition operators, and p_n and $(1 - p_n)$ are, respectively, the probabilities of responses A_1 and A_2 on trial n. A linear model discussed by Bush and Mosteller [8] is obtained when the operators in (1) are defined by the equations

$$\begin{aligned} Q_1 p_n &= \alpha_1 p_n & (0 \le \alpha_1 \le 1), \\ Q_2 p_n &= \alpha_2 p_n & (0 \le \alpha_2 \le 1). \end{aligned} \tag{2}$$

In this paper, this linear model is called the "alpha" model. A specialization of the nonlinear "beta" model proposed by Luce [13] is obtained when the operators are defined by the equations:

$$Q_i p_n = \frac{\beta_i p_n}{1 + (\beta_i - 1)p_n}, \qquad i = 1, 2; \qquad \beta_i > 0; \qquad 0 \ne p_1 \ne 1. \tag{3}$$

In terms of the variable $v_n = p_n/(1 - p_n)$ the transition equations for this version of the beta model are

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*Abstracted from portions of the author's doctoral dissertation, University of Pennsylvania, June 1960. The author is indebted to Robert R. Bush, his dissertation supervisor for the valuable help and encouragement received from him and to R. Duncan Luce for many helpful discussions and for partial support from an NSF grant.

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$$v_{n+1} = \begin{cases} \beta_1 v_n & \text{with probability } p_n \\ \beta_2 v_n & \text{with probability } (1 - p_n), \end{cases} \tag{4}$$

where

$$0 < v < \infty; \qquad \beta_i > 0; \qquad i = 1, 2.$$

In the beta model response probabilities undergo nonlinear rather than linear transformations from trial to trial. Since the probabilities of choice inevitably enter into the derivation of stochastic properties of the model, the methods generally used to derive properties of linear learning models do not apply to the beta model.

Analytical methods applicable to both the alpha and beta models are presented in this paper. The approach used is to consider the branching process defined by the decision rules of the two models, and from it to formulate functional equations for various statistics of interest. Tatsuoka and Mosteller [15] used a functional equation approach to obtain some statistics for the alpha model. Their techniques differ somewhat from those presented here; the approach developed here leads to a unified method of attack for the alpha and beta models and can be extended to others.

Some Random Walks Arising from the Beta Model

In (4), $\beta_i > 1$ and $\beta_i < 1$ may be identified, respectively, with reward and nonreward of the response. If response A_1 is never rewarded and response A_2 is always rewarded $\beta_1 < 1$, $\beta_2 < 1$. If both responses are always rewarded $\beta_1 > 1$, $\beta_2 < 1$. If neither response is ever rewarded $\beta_1 < 1$, $\beta_2 > 1$. It is shown in [11] that these three cases lead to one-absorbing-barrier (OAB), two-absorbing-barrier (TAB), and two-reflecting-barrier (TRB) walks. Rigorous proof of the nature of the barriers for these and other random walks resulting from the two-alternative, two-outcome beta model is given by Lamperti and Suppes [12]. Only the OAB beta model ($\beta_1 \leq 1$, $\beta_2 \leq 1$) is considered in this paper. Except for the case when $\alpha_i = 1$, in the alpha model either response diminishes the probability of response A_1; the alpha model is a one-absorbing-barrier model.

Functional Equations for Statistics of the One-Absorbing Barrier Models

The OAB alpha and beta models lead to an asymptotic distribution of p_n which has all its density at $p = 0$. (Considering response A_1 as an error on the part of organisms which are learning, this means that all organisms eventually learn not to make errors). Additional information about the processes is obtained from various statistics. Following the work of Bush and Sternberg [9] on a simple single-operator model, the statistics considered are those which describe the rate of approach to the asymptote, such as the mean, weighted mean, and variance of the rate of approach; sequential statistics concerning runs of responses; other statistics, such as those describing the first occurrence of an A_2 response (success) and the last occurrence of an A_1 response (failure). Functional equations satisfied by these statistics are derived by considering the branching processes shown in Fig. 1.

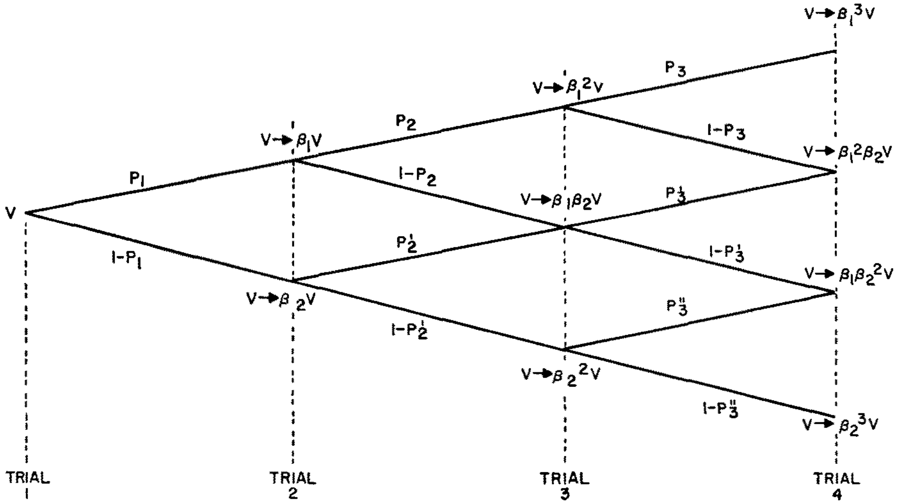
$V \rightarrow \beta_1^3 V$
P_3
$V \rightarrow \beta_1^2 V$
P_2
$V \rightarrow \beta_1 V$
$1-P_3$
$V \rightarrow \beta_1^2 \beta_2 V$
P_1
$1-P_2$
$V \rightarrow \beta_1 \beta_2 V$
P_3'
V
P_2'
$1-P_3'$
$1-P_1$
$V \rightarrow \beta_1 \beta_2^2 V$
$V \rightarrow \beta_2 V$
P_3''
$1-P_2'$
$V \rightarrow \beta_2^2 V$
$1-P_3''$
$V \rightarrow \beta_2^3 V$
TRIAL 1
TRIAL 2
TRIAL 3
TRIAL 4

FIGURE 1a
The Beta Model Lattice

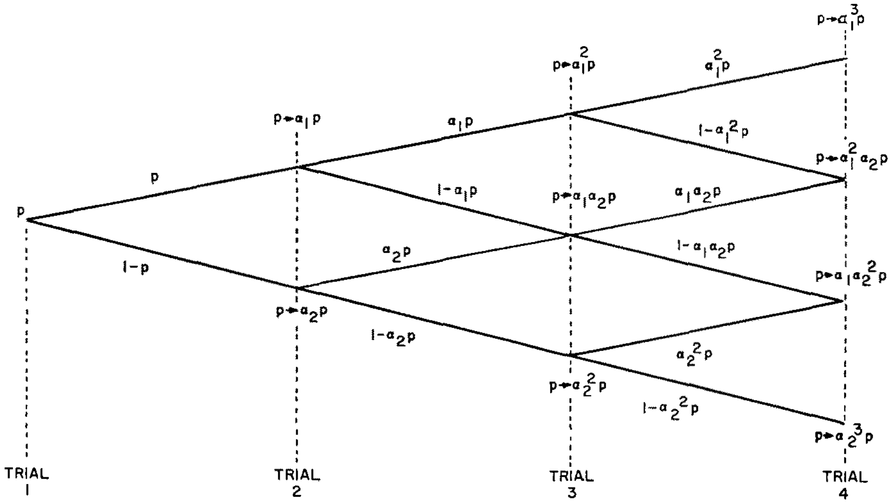
$p \rightarrow \alpha_1^3 p$
$p \rightarrow \alpha_1^2 p$
$\alpha_1^2 p$
$p \rightarrow \alpha_1 p$
$\alpha_1 p$
$1-\alpha_1^2 p$
$p \rightarrow \alpha_1^2 \alpha_2 p$
p
$1-\alpha_1 p$
$p \rightarrow \alpha_1 \alpha_2 p$
$\alpha_1 \alpha_2 p$
p
$\alpha_2 p$
$1-\alpha_1 \alpha_2 p$
$1-p$
$p \rightarrow \alpha_1 \alpha_2^2 p$
$p \rightarrow \alpha_2 p$
$1-\alpha_2 p$
$\alpha_2^2 p$
$p \rightarrow \alpha_2^2 p$
$1-\alpha_2^2 p$
$p \rightarrow \alpha_2^3 p$
TRIAL 1
TRIAL 2
TRIAL 3
TRIAL 4

FIGURE 1b
The Alpha Model Lattice

For the analysis which follows, a sequence $x_1, x_2, \cdots, x_n$ of random variables is defined such that

$$x_n = \begin{cases} 1 \text{ if response } A_1 \text{ occurs on trial } n \\ 0 \text{ if response } A_2 \text{ occurs on trial } n. \end{cases}$$

The random variables have expectations p_n.

The mean number of A_1 responses

In terms of the random variables x_n, the total number of A_1 responses in N trials is given by the random variable $X_N = \sum_{n=1}^{N} x_n$ with expectation $E(X_N) = \sum_{n=1}^{N} p_n$. In the one-absorbing barrier models, both responses decrease the probability of response A_1 and

$$E(X) = \lim_{N \to \infty} E(X_N)$$

is of interest. In fact, by replacing the parameters of the models by $\beta = \max(\beta_1, \beta_2)$ and $\alpha = \max(\alpha_1, \alpha_2)$ finite upper bounds for $E(X)$ in the two models are obtained. Now the number X_N of A_1 responses in N trials starting from trial 1 will be equal to the number, X_{N-1}, of A_1 responses in $(N - 1)$ trials starting from trial 2 if the result of trial 1 is an A_2 response and be equal to $1 + X_{N-1}$ if the result of trial 1 is an A_1 response. Letting ϕ denote the expected number of A_1 responses, the functional equations for ϕ are obtained from Fig. 1 to be

$$\begin{aligned} \phi_\beta(v, N) &= p_1[1 + \phi_\beta(\beta_1 v, N - 1)] + (1 - p_1)\phi_\beta(\beta_2 v, N - 1) \\ &= \frac{v}{1 + v}[\phi_\beta(\beta_1 v, N - 1) + 1] + \frac{1}{1 + v}\phi_\beta(\beta_2 v, N - 1), \end{aligned}$$

and

$$\phi_\alpha(p, N) = p[\phi_\alpha(\alpha_1 p, N - 1) + 1] + (1 - p)\phi_\alpha(\alpha_2 p, N - 1).$$

When $N \to \infty$ these equations become

$$\phi_\beta(v) = \frac{v}{1 + v}\phi_\beta(\beta_1 v) + \frac{1}{1 + v}\phi_\beta(\beta_2 v) + \frac{v}{1 + v}, \tag{5}$$

$$\phi_\alpha(p) = p\phi_\alpha(\alpha_1 p) + (1 - p)\phi_\alpha(\alpha_2 p) + p. \tag{6}$$

Both the above functions must, of course, satisfy the boundary condition $\phi(0) = 0$.

The second moment of the number of A_1 responses

Letting θ denote $E(X_N^2)$ the functional equations for the second moment are then, as $N \to \infty$,

$$\theta_\beta(v) = \frac{v}{1 + v}\theta_\beta(\beta_1 v) + \frac{1}{1 + v}\theta_\beta(\beta_2 v) + \frac{v}{1 + v}[1 + 2\phi_\beta(\beta_1 v)], \tag{7}$$

$$\theta_\alpha(p) = p\theta_\alpha(\alpha_1 p) + (1 - p)\theta_\alpha(\alpha_2 p) + p[1 + 2\phi_\alpha(\alpha_1 p)]. \tag{8}$$

$\theta(0) = 0$ is a boundary condition. Finite upper bounds exist for $\theta_\beta(v)$ and $\theta_\alpha(p)$; replacing the parameters by $\beta = \max(\beta_1, \beta_2)$ and $\alpha = \max(\alpha_1, \alpha_2)$ the variance of X_N is $\sum_{n=1}^{N} p_n(1 - p_n)$ which remains finite as $N \to \infty$ if $\sum_1^N p_n$ does. Functional equations for higher moments are easily obtained in this manner.

The functional equations for the mean and second moment of the number of A_1 responses have been previously obtained by Tatsuoka [14] and Tatsuoka and Mosteller [15]. Their method of derivation is somewhat different from that presented here.

The weighted number of A_1 responses

Define the random variable

$$Y_{\omega,N} = \sum_{n=1}^{N} (n + \omega)x_n .$$

Then $Y_{0,N}$ represents the weighted number of A_1 responses in N trials with the weighting function being the trial number n. From trial 2 on, the weighted number of A_1 responses is $\sum_{n=2}^{N} nx_n$, which by relabeling the random variables $x_2, x_3, \cdots$ as $x_1, x_2, \cdots$ can be represented by the random variable

$$Y_{1,(N-1)} = \sum_{n=1}^{N-1} (n + 1)x_n = Y_{0,N-1} + X_{N-1} .$$

If ψ stands for the expectation of the weighted number of A_1 responses, the functional equations are obtained by noting that $Y_{0,N}$ is equal to $Y_{1,N-1}$ if the result of the first trial is an A_2 response and is equal to $(1 + Y_{1,(N-1)})$ if the result of the first trial is an A_1 response. For an infinite number of trials,

$$\psi_\beta(v) = \frac{v}{1 + v}\, \psi_\beta(\beta_1 v) + \frac{1}{1 + v}\, \psi_\beta(\beta_2 v) + \phi_\beta(v), \tag{9}$$

$$\psi_\alpha(p) = p\psi_\alpha(\alpha_1 p) + (1 - p)\psi(\alpha_2 p) + \phi_\alpha(p). \tag{10}$$

A boundary condition is $\psi(0) = 0$.

Number of trials before the first A_2 response (success) occurs

Let $F_1 + 1$ denote the trial number on which response A_2 occurs for the first time so that F_1 is the number of trials before the first A_2. F_1 is equal to zero if A_2 occurs on the first trial and is equal to $(1 + F_2)$, where F_2 denotes the number of trials, before the first A_2 response occurs, starting at trial 2, if trial 1 results in an A_1 response. Letting ν denote the expectation of the random variables F, the functional equations for ν are

$$\nu_\beta(v) = p_1[\nu_\beta(\beta_1 v) + 1] + [(1 - p_1)0] = \frac{v}{1 + v}\, \nu_\beta(\beta_1 v) + \frac{v}{1 + v}\, ; \tag{11}$$

$$\nu_\alpha(p) = p\nu_\alpha(\alpha_1 p) + p. \tag{12}$$

If ρ denotes the second moment of the random variables F, the functional equations for ρ are

$$\begin{aligned} \rho_\beta(v) &= \frac{v}{1+v}\,[1 + 2\nu_\beta(\beta_1 v) + \rho_\beta(\beta_1 v)] \\ &= \frac{v}{1+v}\,\rho_\beta(\beta_1 v) + \frac{v}{1+v}\,[1 + 2\nu_\beta(\beta_1 v)]; \end{aligned} \tag{13}$$

$$\rho_\alpha(p) = p\rho_\alpha(\alpha_1\ , p) + p[1 + 2\nu(\alpha_1 p)]. \tag{14}$$

Trial number at which last A_1 response occurs

Let

$$L_n = \begin{cases} 0 \text{ if no } A_1 \text{ response occurs on, or after trial } n \\ 1 \text{ if the last } A_1 \text{ response occurs on trial } n \\ (N+1) - n \text{ if the last } A_1 \text{ response occurs on trial } N > n. \end{cases}$$

Then the random variable L_1 represents the trial number at which the last A_1 response occurs, and by definition L_1 is zero if no A_1 response occurs on any trial. In the following development, the sequence of responses $A_2A_2A_1$ denotes the occurrence of A_2 on the first trial followed by A_2 on the second trial and by A_1 on the third trial. It is evident that

$$L_1 = \begin{cases} L_2 + 1 & \text{if } A_1 \text{ occurs} \\ L_3 + 2 & \text{if } A_2A_1 \text{ occurs} \\ L_4 + 3 & \text{if } A_2A_2A_1 \text{ occurs} \\ L_5 + 4 & \text{if } A_2A_2A_2A_1 \text{ occurs} \\ \text{and so on.} \end{cases}$$

Letting μ denote the expectation of the random variables L, the functional equation for μ_α is developed from Fig. 1. For an infinite number of trials

$$\begin{aligned} \mu_\alpha(p) &= p[\mu_\alpha(\alpha_{1p}) + 1] + (1-p)\alpha_2 p[\mu_\alpha(\alpha_1\alpha_2 p) + 2] \\ &\qquad + (1-p)(1-\alpha_2 p)\alpha_2^2 p[\mu_\alpha(\alpha_1\alpha_2^2 p) + 3] + \cdots \\ &= p\mu_\alpha(\alpha_1 p) + p + (1-p)\alpha_2 p + (1-p)(1-\alpha_2 p)\alpha_2^2 p + \cdots \\ &\qquad + [(1-p)\alpha_2 p + 2(1-p)(1-\alpha_2 p)\alpha_2^2 p \\ &\qquad + 3(1-p)(1-\alpha_2 p)(1-\alpha_2^2 p)\alpha_2^3 p + \cdots + (1-p)\alpha_2 p\mu_\alpha(\alpha_1\alpha_2 p) \\ &\qquad + (1-p)(1-\alpha_2 p)\alpha_2^2 p\mu_\alpha(\alpha_1\alpha_2^2 p) + \cdots]. \end{aligned}$$

But the term in brackets in the last expression is just $(1-p)\mu_\alpha(\alpha_2 p)$ as may be deduced from the expression for $\mu_\alpha(p)$. Also

$$p + (1-p)\alpha_2 p + (1-p)(1-\alpha_2 p)\alpha_2^2 p + \cdots = 1 - \prod_{i=0}^{\infty}(1-\alpha_2^i p).$$

A similar development for $\mu_\beta(v)$ results in the functional equations

$$(15)\qquad \mu_\beta(v) = \frac{v}{1+v}\,\mu_\beta(\beta_1 v) + \frac{1}{1+v}\,\mu_\beta(\beta_2 v) + 1 - \prod_{i=0}^{\infty} \frac{1}{(1+\beta_2^i v)},$$

$$(16)\qquad \mu_\alpha(p) = p\mu_\alpha(\alpha_1 p) + (1-p)\mu_\alpha(\alpha_2 p) + 1 - \prod_{i=0}^{\infty} (1-\alpha_2^i p),$$

with $\mu(0) = 0$, since for $p = 0$ no A_1 response ever occurs. A different derivation for the mean of the trial number at which the last A_1 response occurs in the alpha model will be found in Tatsuoka and Mosteller [15].

For the expectation of L_1^2 it is necessary to consider the expectations of $(L_2 + 1)^2$, etc. Denoting the second moment of the random variables L by γ the functional equations for γ are

$$(17)\quad \gamma_\beta(v) = \frac{v}{1+v}\gamma_\beta(\beta_1 v) + \frac{1}{1+v}\gamma_\beta(\beta_2 v) + \left[2\mu_\beta(v) + \prod_{i=0}^{\infty}\frac{1}{1+\beta_2^i v} - 1\right],$$

and

$$(18)\quad \gamma_\alpha(p) = p\gamma_\alpha(\alpha_1 p) + (1-p)\gamma_\alpha(\alpha_2 p) + \left[2\mu_\alpha(p) + \prod_{i=0}^{\infty}(1-\alpha_2^i p) - 1\right],$$

with $\gamma(0) = 0$. Functional equations for higher moments of L_1 can easily be generated in the above manner.

Number of runs, of length j, of A_1 responses

The sequence of responses

$$\underbrace{A_1 A_1 A_1 \cdots A_1}_{j \text{ trials}} A_2$$

is termed a run, of A_1 responses, of length j. Statistics concerning the number of runs of A_1 responses of length exactly equal to j, and of length greater than or equal to j ($j = 1, 2 \cdots$), are of interest. Let $R_{n,j}$ denote the number of runs of length j, which occur between trial n and the termination of the process. The total number of runs of length j is then $R_{1,j}$. From the branching process of Fig. 1 it is seen that $R_{1,j} = R_{n,j} + \delta_{n,j+2}$, where $\delta_{n,j+2}$ is the Kronecker delta function. Letting σ_j denote the expectation of the number of runs of length j, the functional equation for $\sigma_{j\beta}$ is developed from the beta model lattice of Fig. 1(a). For an infinite number of trials

$$\sigma_{j\beta}(v) = \sum_{k=0}^{\infty}\left(\prod_{i=1}^{k} p_i\right)(1 - p_{k+1})[\sigma_{j\beta}(\beta_1^k\beta_2 v) + \delta_{k,j}],$$

where $\delta_{k,j}$ is the Kronecker delta function. Substituting $\sigma_{j\beta}(\beta_1 v)$ for part of the expression gives,

$$\sigma_{j\beta}(v) = p_1\sigma_{j\beta}(\beta_1 v) + (1-p_1)\sigma_{j\beta}(\beta_2 v) + \prod_{i=1}^{j} p_i[(1-p_{j+1}) - p_{j+1}(1-p_{j+2})].$$

A similar development gives the functional equation for $\sigma_{j\alpha}(p)$. The functional equations are

$$(19) \qquad \sigma_{j\beta}(v) = \frac{v}{1+v}\,\sigma_{j\beta}(\beta_1 v) + \frac{1}{1+v}\,\sigma_{j\beta}(\beta_2 v) + \prod_{i=1}^{j+1}\left(\frac{\beta_1^{i-1}v}{1+\beta_1^{i-1}v}\right)\left(\frac{1}{\beta_1^{j}v} - \frac{1}{1+\beta_1^{j+1}v}\right);$$

$$(20) \qquad \sigma_{j\alpha}(p) = p\sigma_{j\alpha}(\alpha_1 p) + (1-p)\sigma_{j\alpha}(\alpha_2 p) + \alpha_1\,\frac{j(j-1)}{2}\,p^{j}(1 - 2\alpha_1^{j}p + \alpha_1^{2j+1}p^2),$$

with $\sigma_j(0) = 0$.

Number of runs of length greater than or equal to j

Let $T_{n,j}$ stand for the number of runs of length greater than or equal to j of A_1 responses, which occur from trial n to the termination of the process. Then $T_{1,j}$ denotes the total number of such runs. Now, for an infinite number of trials

$$T_{1,j} = T_{n,j} + \delta_{n,j+i} \qquad (i = 2, 3, \cdots).$$

Letting λ_j be the expectation of the number of runs of length $\geq j$, a development similar to that previously outlined gives

$$(21) \qquad \lambda_{j\beta}(v) = p_1\lambda_{j\beta}(\beta_1 v) + (1-p_1)\lambda_{j\beta}(\beta_2 v) + (1-p_{j+1})\prod_{i=1}^{j+1} p_i$$
$$= \frac{v}{1+v}\,\lambda_{j\beta}(\beta_1 v) + \frac{1}{1+v}\,\lambda_{j\beta}(\beta_2 v) + \frac{1}{\beta_1^{j}v}\prod_{i=1}^{j+1}\frac{\beta_1^{i-1}v}{1+\beta_i^{i-1}v},$$

$$(22) \qquad \lambda_{j\alpha}(p) = p\lambda_{j\alpha}(\alpha_1 p) + (1-p)\lambda_{j\alpha}(\alpha_2 p) + (1-\alpha_1^{j}p)\alpha_1\,\frac{j(j-1)}{2}\,p^{j}.$$

The expectation of the total number of runs of A_1 responses in an infinite number of trials is obtained when $j = 1$. Denoting this statistic by λ,

$$(23) \qquad \lambda_\beta(v) = \frac{v}{1+v}\,\lambda_\beta(\beta_1 v) + \frac{1}{1+v}\,\lambda_\beta(\beta_2 v) + \frac{v}{1+v}\cdot\frac{1}{(1+\beta_1 v)},$$

$$(24) \qquad \lambda_\alpha(p) = p\lambda_\alpha(\alpha_1 p) + (1-p)\lambda_\alpha(\alpha_2 p) + p(1-\alpha_1 p),$$

with $\lambda(0) = 0$. Additional functional equations for other random variables of interest, such as runs of A_2 responses, have been derived in [11].

General Functional Equations for the One-Absorbing-Barrier Models

The functional equations presented for statistics of the beta model have the general form

$$f(v, \beta_1, \beta_2) = \frac{v}{1+v} f(\beta_1 v, \beta_1, \beta_2) + \frac{1}{1+v} f(\beta_2 v, \beta_1, \beta_2) + g(v, \beta_1, \beta_2) \tag{25}$$

where

$$0 \leq v < \infty, \qquad 0 \leq \beta_1 \leq 1, \qquad 0 \leq \beta_2 \leq 1.$$

The term $g(v, \beta_1, \beta_2)$ is, in general, different for each statistic considered. For all except the run statistics

$$\begin{gathered} g(v, \beta_1, \beta_2) \geq 0, \\ g(0, \beta_1, \beta_2) = 0, \\ \lim_{v \to \infty} g(v, \beta_1, \beta_2) \geq 1. \end{gathered} \tag{26}$$

For these statistics

$$\begin{gathered} f(0, \beta_1, \beta_2) = 0, \\ \lim_{v \to \infty} f(v, \beta_1, \beta_2) = \infty. \end{gathered} \tag{27}$$

Equation (26) does not hold for the run statistics and the boundary conditions for the run statistics have to be defined separately.

The functional equations for statistics of the alpha model are seen to have the general form

$$y(p, \alpha_1, \alpha_2) = p y(\alpha_1 p, \alpha_1, \alpha_2) + (1 - p) y(\alpha_2 p, \alpha_1, \alpha_2) + z(p, \alpha_1, \alpha_2) \tag{28}$$

where

$$0 \leq p \leq 1, \qquad 0 \leq \alpha_1 \leq 1, \qquad 0 \leq \alpha_2 \leq 1.$$

For the statistics of the alpha model

$$\begin{gathered} z(0, \alpha_1, \alpha_2) = 0, \\ z(1, \alpha_1, \alpha_2) \geq 0, \end{gathered} \tag{29}$$

and the boundary conditions for all the statistics considered are

$$y(0, \alpha_1, \alpha_2) = 0 \tag{30}$$

and

$$\lim_{p \to 1} y(p, \alpha_1, \alpha_2) \quad \text{is finite.}$$

The functional equations for the run statistics of the beta model differ in nature from the functional equations for the other statistics considered. A discussion of the functional equations for the run statistics is presented in [11].

The sections which follow present formal solutions to (25) and (28) under the boundary conditions (27) and (30) respectively. Theorems concerning existence, uniqueness and other properties of the solutions have been proved in [11] by methods similar to those of Bellman [3]. Some of these theorems are stated here without proof.

On the functional equation for the OAB beta model

Writing $f(v, \beta_1, \beta_2)$ simply as $f(v)$, (25) takes the form

$$f(v) = \frac{v}{1+v} f(\beta_1 v) + \frac{1}{1+v} f(\beta_2 v) + g(v), \tag{31}$$

where

$$g(v) \geq 0, \qquad g(0) = 0, \qquad \lim_{v \to \infty} g(v) \geq 1,$$

and

$$f(0) = 0; \qquad \lim_{v \to \infty} f(v) = \infty.$$

Further, let $0 \leq \beta_1 < 1; 0 \leq \beta_2 < 1$. The cases ($\beta_1 = 1, \beta_2 < 1$) and ($\beta_1 < 1, \beta_2 = 1$) can be considered separately.

Existence of solution. For any function $r(v)$ define the operator T by

$$T \cdot r(v) = \frac{v}{1+v} r(\beta_1 v) + \frac{1}{1+v} r(\beta_2 v) + g(v). \tag{32}$$

THEOREM 1.

$$f(v) = \lim_{n \to \infty} T^{(n)} g(v)$$

when the limit exists.

THEOREM 2. *If $g(v)$ is a monotone increasing function of v, then a solution $f(v)$ exists if*

$$\sum_{i=0}^{\infty} g(\beta^i v)$$

is finite for $0 \leq v < \infty$, where $0 \leq \beta = \max(\beta_1, \beta_2) < 1$.

As almost all the $g(v)$ occurring in the beta model first-moment equations are monotone increasing functions of v which satisfy the conditions of Theorem 2, the existence of the mean of most of the random variables introduced for the OAB beta model is assured.

From a proof similar to that for Theorem 2 it follows that when $g(v)$ is a monotone increasing function of v,

$$\sum_{i=0}^{\infty} g(\beta_n^i v) \leq f(v) \leq \sum_{i=0}^{\infty} g(\beta_m^i v), \tag{33}$$

where

$$\beta_m = \max(\beta_1, \beta_2) \quad \text{and} \quad \beta_n = \min(\beta_1, \beta_2).$$

Continuity. If $| g(v) |$ is bounded in $0 \leq v < \infty$, the solution $f(v)$ is continuous.

Monotonicity. If $g(v)$ is a monotone increasing function of v, and if $\beta_1 \geq \beta_2$, then $f(v)$ is a monotone increasing function of v.

Uniqueness. The solution $f(v)$ is unique in $0 \leq v < \infty$.

On the functional equation for the OAB alpha models

For the functional equation

$$y(p) = p \cdot y(\alpha_1 p) + (1 - p) y(\alpha_2 p) + z(p) \tag{34}$$

the development of existence, uniqueness and other properties of the solution is similar to that for (31). Some properties of $y(p)$ are stated without proof.

Existence. For any function $Q(p)$, define the operator

$$\Lambda Q(p) = pQ(\alpha_1 p) + (1 - p) Q(\alpha_2 p) + z(p) \tag{35}$$

and let

$$\lim_{n \to \infty} \Lambda^{(n)} \cdot z(p) \,|_{p \to 1} = c(\alpha_1, \alpha_2).$$

THEOREM 3.

$$y(p) = \lim_{n \to \infty} \Lambda^{(n)} \cdot z(p).$$

THEOREM 4. *If $z(p)$ is monotone increasing in p, then*

$$\sum_{i=0}^{\infty} z(\alpha_n^i p) \leq y(p) \leq \sum_{i=0}^{\infty} z(\alpha_m^i p)$$

where

$$\alpha_m = \max(\alpha_1, \alpha_2), \qquad \alpha_n = \min(\alpha_1, \alpha_2).$$

Monotonicity. If $z(p)$ is monotone increasing in p, and $\alpha_1 \geq \alpha_2$, then $y(p)$ is monotone increasing in p.

Convexity. If $z(p)$ is convex and $\alpha_1 \geq \alpha_2$, then $y(p)$ is convex.

Solution of the Functional Equation for the OAB Beta Model

The solution to (31) is obtained by generalizing from solutions of the equation for special parameter values. The parameter space of the OAB beta model is shown in Fig. 2. One solution for special parameter values is derived here. A detailed presentation will be found in [11].

THEOREM 5. *Along sides* (1) *and* (2) *of Fig.* 2,

$$f(v) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} g(\beta_1^m \beta_2^n v) \prod_{j=0}^{m-1} \frac{\beta_2^n \beta_1^j v}{(1 + \beta_2^n \beta_1^j v)}. \tag{36}$$

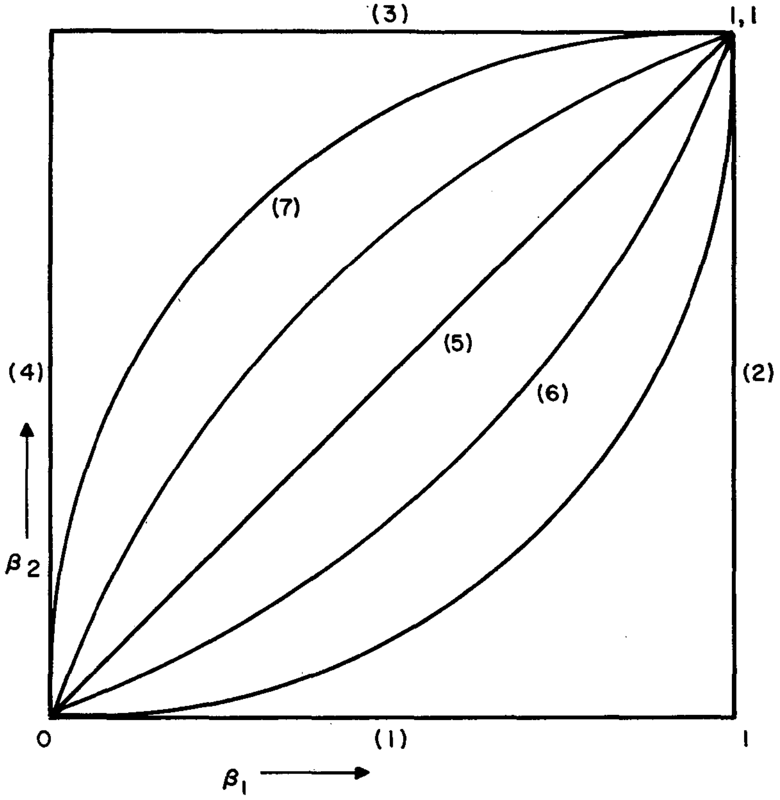

FIGURE 2
Parameter Space of OAB Beta Model

PROOF. Along side (1), $\beta_2 = 0$, $\beta_1 \leq 1$, and only the $n = 0$ term of the summation over n, is nonzero. The resulting expression is the one obtained from the functional equation, for in this case

$$f(v) = \frac{v}{1+v} f(\beta_1 v) + g(v),$$

giving

$$f(\beta_1^m v) = \frac{\beta_1^m v}{(1+\beta_1^m v)} f(\beta_1^{m+1} v) + g(\beta_1^m v),$$

for $m = 0, 1, \cdots$, from which the desired result is obtained by successive substitution. Along side (2), $\beta_1 = 1$, $\beta_2 \leq 1$, and (36) becomes

$$\sum_{n=0}^{\infty} g(\beta_2^n v) \sum_{m=0}^{\infty} \prod_{j=0}^{m-1} \frac{\beta_2^n v}{(1+\beta_2^n v)} = \sum_{n=0}^{\infty} (1+\beta_2^n v) g(\beta_2^n v),$$

the last expression being also the one obtained from the functional equation for the case $\beta_1 = 1$, $\beta_2 < 1$, for which case the functional equation reduces to $f(v) - f(\beta_2 v) = (1 + v)g(v)$. Q.E.D.

Note that at the point (1, 1) of the parameter space the solutions diverge. By letting $\beta_1 = \beta_2^k$, $\beta_2 = \beta_1^k$ $(k = 1, 2, \cdots)$, solutions along arcs of the form (6) and (7) of the parameter space can be obtained. The resulting functional equations may be written in the form of q-difference equations for which there exists an extensive body of literature [1, 2].

Examination of the solutions for various special parameter values suggests the form of the general solution. The general solution to (31) is given by the following theorem.

THEOREM 6.

$$f(v) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{m,n}(v) \cdot g(\beta_1^m \beta_2^n v).$$

where

$$A_{0,0}(v) = 1,$$

$$A_{m,0}(v) = \prod_{i=1}^{m} \frac{\beta_1^{i-1} v}{1 + \beta_1^{i-1} v} \qquad (m = 1, 2, \cdots),$$

$$A_{0,n}(v) = \prod_{j=1}^{n} \frac{1}{1 + \beta_2^{j-1} v} \qquad (n = 1, 2, \cdots),$$

$$A_{m,n}(v) = \sum_{k=0}^{n} A_{0,k}(v) A_{1,0}(\beta_2^k v) A_{m-1,n-k}(\beta_1 \beta_2^k v) \qquad (m, n = 1, 2, \cdots).$$

PROOF. Substitution in (31) gives

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{m,n}(v) g(\beta_1^m \beta_2^n v) = \frac{v}{1 + v} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{m,n}(\beta_1 v) g(\beta_1^{m+1} \beta_2^n v) + \frac{1}{1 + v} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{m,n}(\beta_2 v) g(\beta_1^m \beta_2^{n+1} v) + g(v)$$

so that

$$\sum_{m=0}^{\infty} A_{m,0}(v) g(\beta_1^m v) = g(v) + \frac{v}{1 + v} \sum_{m=1}^{\infty} A_{m-1,0}(\beta_1 v) g(\beta_1^m v), \tag{37}$$

which gives

$$A_{0,0}(v) = 1; \qquad A_{m,0}(v) = \frac{v}{1 + v} A_{m-1,0}(\beta_1 v) = \prod_{i=1}^{m} \frac{\beta_1^{i-1} v}{1 + \beta_1^{i-1} v}.$$

$$\sum_{n=1}^{\infty} A_{0,n}(v) g(\beta_2^n v) = \frac{1}{1 \mp v} \sum_{n=1}^{\infty} A_{0,n-1}(\beta_2 v) g(\beta_2^n v), \tag{38}$$

which gives,

$$A_{0,1}(v) = \frac{1}{1+v} A_{0,0}(\beta_2 v) = \frac{1}{1+v},$$

$$A_{0,n}(v) = \frac{1}{1+v} A_{0,n-1}(\beta_2 v) = \prod_{j=1}^{n} \frac{1}{(1+\beta_2^{j-1} v)},$$

and

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{m,n}(v) g(\beta_2^m \beta_2^n v) = \frac{v}{1+v} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{m-1,n}(\beta_1 v) g(\beta_1^m \beta_2^n v) + \frac{1}{1+v} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{m,n-1}(\beta_2 v) g(\beta_1^m \beta_2^n v). \tag{39}$$

The coefficients in this last expression satisfy the difference equation

$$A_{m,n}(v) = \frac{v}{1+v} A_{m-1,n}(\beta_1 v) + \frac{1}{1+v} A_{m,n-1}(\beta_2 v),$$

from which follows [11]

$$A_{1,n}(v) = \sum_{k=0}^{n} A_{0,k}(v) A_{1,0}(\beta_2^k v) A_{0,n-k}(\beta_1 \beta_2^k v)$$

$$= \sum_{k=0}^{n} \frac{\beta_2^k v}{1+\beta_2^k v} \prod_{j=1}^{k} \frac{1}{(1+\beta_2^{j-1} v)} \prod_{j=1}^{n-k} \frac{1}{(1+\beta_1 \beta_2^{j+k-1} v)},$$

$$A_{m,n}(v) = \sum_{k=0}^{n} A_{0,k}(v) A_{1,0}(\beta_2^k v) A_{m-1,n-k}(\beta_1 \beta_2^k v). \qquad \text{Q.E.D.}$$

General Solution to the OAB Alpha Model Functional Equation

Replacing β_2 and β_1 by α_2 and α_1 in Fig. 2 gives the parameter space of the OAB alpha model. The general solution for (34) can be derived [11] in a manner similar to that used for the beta model functional equation. The solution is given by

THEOREM 7.

$$y(p) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} b_{m,n}(p) \cdot z(\alpha_1^m \alpha_2^n p),$$

where

$$b_{0,0}(p) = 1,$$

$$b_{m,0}(p) = p^m \alpha_1 \frac{m(m-1)}{2} \qquad (m = 1, 2, \cdots),$$

$$b_{0,n}(p) = \prod_{j=1}^{n} (1 - \alpha_2^{j-1} p) \qquad (n = 1, 2, \cdots),$$

$$b_{m,n}(p) = \sum_{k=0}^{n} b_{1,0}(\alpha_2^k p) b_{0,k}(p) b_{m-1,n-k}(\alpha_1 \alpha_2^k p) \qquad (m, n = 1, 2, \cdots).$$

PROOF. The proof is similar to that used for the beta model equation. Details are given in [11].

Discussion

Analytical techniques applicable to a class of learning models have been presented. Functional equations for various statistics of two learning models, viz., Luce's nonlinear beta model and a linear commuting-operator model called the alpha model, have been derived from the branching processes defined by the models.

The results on stochastic properties of Luce's beta model are new. For the alpha model, power series solutions to the functional equations for the first and second moments of the total number of A_1 responses and the trial number at which the last A_1 occurs had been obtained by Tatsuoka and Mosteller [15]. However, the techniques of expanding the functions in a power series in the variable often fails, as is illustrated by the fact that the power series solutions (obtained by Tatsuoka [14]) to the functional equations for the first and second moments of the total number of A_1 responses for the one-absorbing-barrier (OAB) beta model are not valid for $v \geq 1$.

By investigating two general equations, the problem of solving the individual functional equations for the OAB models was simplified. The functional equations for the sequential statistics of the OAB beta model do not have the same boundary conditions as the general equation presented in this paper, and their solutions require additional investigation.

Because of the complexity of the expressions obtained for the statistics of the OAB models, an attempt was made to find some close bounds which could be easily computed. Some upper and lower bounds for statistics of the OAB alpha model have been presented in ([11], ch. 5). An upper bound for one statistic of the OAB beta model has also been derived in [11], mainly to illustrate the methods used to obtain upper bounds for a few statistics of the OAB beta model. These methods failed for a number of the statistics. Furthermore, a method for the derivation of close lower bounds for the OAB beta model remains to be found.

Empirical tests and comparisons of the beta model with other models have been presented by Bush, Galanter, and Luce [6] and Fey [10]. The use of statistics such as those derived in this paper for the estimation of parameters and for measuring the goodness of fit has been discussed by Bush and Mosteller [8], Bush, Galanter, and Luce [6] and by others (see [5]).

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Manuscript received 12/3/60

Revised manuscript received 7/23/61